

Gaussian Approach to (1 + 1) Dimensional Solitons at Finite Temperature

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Soliton solutions at finite temperature have been studied using the Gaussian effective potential approach. A critical temperature T_c is obtained at which the soliton solutions cease to exist. The effective potential has also been calculated together with the Mass Gap equation.

1. Introduction

Finite temperature field theory has been studied by many authors since the publications of the original works of Dolan and Jackiw [1] and Weinberg [2]. However not many studies have been made of solitons at finite temperature. Aoyama and Quinn [3] investigated the non-perturbative correction to finite temperature field theory in a scalar real field. Recently Su et al. [4, 5] studied the soliton solutions of the φ^4 and φ^6 fields at finite temperature using the method of coherent state and the real time Green function formalism. Kuczyński and Manka [6] also studied the soliton solution in the $\varphi^3 + \varphi^4$ field using Bogoliubov inequality. In this paper we have studied soliton solutions of the (1 + 1) dimensional real φ^4 field at finite temperature using the Gaussian effective potential (GEP) method which has been successfully applied to the study of φ^4 field both in flat and curved space time (Stevenson [7], Alles and Tarrach [8], Koniuk and Tarrach [9]). Our method differs from that of Su et al. [10] in the following respect. Their effective potential is different from that of ours in the sense that they did not consider the contribution of kinetic term. This difference will be discussed later on. Also, we think their normalization for effective mass is not rigorous. Roditi [11] has also considered finite temperature φ^4 theory. However Roditi has not considered the 1 + 1 dimensional soliton and in any case has not given the mass gap condition. The GEP method has a certain advantage over other approaches. It is known to contain the one loop and the $1/N$ result as limiting cases. As will be shown here, this method coupled with the

zeta function regularization method (Roy and Roychoudhury [12]) based on imaginary time formalism [1] can easily be extended to the study of solitons at finite temperature. This method involves very simple and straight forward calculations.

2. Finite Temperature Effective Potential for $\lambda\varphi^4$ in (1 + 1) Dimension

Let us start with the Lagrangian density of the (1 + 1) dimensional φ^4 model,

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{m_b^2}{4} \varphi^2 - \frac{\lambda}{4} \varphi^4, \quad (1)$$

where the notations are standard [13]. At the classical level, Eq.(1) yields a static Kink solution given by Rajaraman [13] ($\bar{m}_b = m_b/\sqrt{2}$)

$$\varphi(x) = \pm \left(\frac{\bar{m}_b}{\sqrt{\lambda}} \right) \tanh \left(\frac{\bar{m}_b}{\sqrt{2}} (x - x_0) \right). \quad (2)$$

The energy density of the Kink is localised with width of the order of $1/m$. In order to use the Gaussian variational method we introduce the ansatz (in $v+1$ dimension)

$$\varphi(x) = \varphi_0 + \int \frac{d^v k}{(2\pi)^v w_k(\Omega^2)} [a_\Omega(\mathbf{k}) e^{-ikx} + a_\Omega^\dagger(\mathbf{k}) e^{ikx}], \quad (3)$$

where

$$w_k(\Omega^2) = \sqrt{\Omega^2 + k^2}. \quad (4)$$

Both φ_0 and Ω^2 will be treated as variational parameters. The annihilation and creation operators a_Ω and a_Ω^\dagger satisfy the following commutation relation:

$$[a_\Omega(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^v 2 w_k(\Omega) \delta^v(\mathbf{k} - \mathbf{k}'). \quad (5)$$

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According to the variational principle, the upper bound of the ground state energy is given by the minimum of $\langle O_\Omega | H | O_\Omega \rangle$, where H is the Hamiltonian density of the system and $|O\rangle_\Omega$, φ_0 is a normalized Gaussian wave function centered on $\varphi = \varphi_0$. We write

$$V^G(\varphi_0, \Omega(\varphi_0)) = \langle O_\Omega | H | O_\Omega \rangle. \quad (6)$$

To calculate $\langle O_\Omega | H | O_\Omega \rangle$ we use the following results ([7], 1984)

$$\langle O | \frac{1}{2} \varphi^2 + \frac{1}{2} (\bar{V}\varphi)^2 | O \rangle = I_1^\beta - \frac{\Omega^2}{2} I_0^\beta, \quad (7)$$

$$\langle O | \varphi^2 | O \rangle = \varphi_0^2 + I_0^\beta, \quad (8)$$

$$\langle O | \varphi^4 | O \rangle = \varphi_0^4 + 6 \varphi_0^2 I_0^\beta + 3 I_0^{\beta^2}. \quad (9)$$

Using the above results we obtain from (6)

$$V^G(\varphi_0, \Omega) = I_1^\beta - \frac{\Omega^2}{2} I_0^\beta + \frac{\lambda}{4} (\varphi_0^4 + 6 \varphi_0^2 I_0^\beta + 3 I_0^{\beta^2}) - \frac{m_b^2}{4} (\varphi_0^2 + I_0^\beta), \quad (10)$$

where I_n^β is given by

$$I_n^\beta = \int \frac{d^v k}{(2\pi)^v 2 w_k(\Omega^2)} [w_k(\Omega^2)]^n. \quad (11)$$

The integral (11) is to be calculated taking finite temperature into account. For zero temperature, $I_n(\equiv I_n^\beta(T=0))$ satisfies

$$\frac{dI_n}{d\Omega} = (2n-1) I_{n-1}(\Omega^2), \quad n=0, 1, \dots \quad (12)$$

Hazi and Stevenson [14] and Roditi [11] have shown that, if one uses a covariant formalism, then (12) also holds at finite temperature. Now the optimum value $\bar{\Omega}$ of Ω is given by

$$dV^G/d\Omega = 0|_{\Omega=\bar{\Omega}}. \quad (13)$$

Using (10) and (12) we get

$$\bar{\Omega}^2 = 3\lambda(I_0^\beta + \varphi_0^2) - m_b^2/2. \quad (14)$$

Using (14), $V^G(\varphi_0)$ is now given by

$$\bar{V}_G(\varphi_0) = I_1^\beta - \frac{3\lambda}{4} (I_0^\beta)^2 + \frac{\lambda}{4} \varphi_0^4 - \frac{m_b^2}{4} \varphi_0^2. \quad (15)$$

From (15) we get

$$\frac{d\bar{V}_G(\varphi_0)}{d\varphi_0} = \varphi_0 \left(-\frac{m_b^2}{2} + \lambda \varphi_0^2 + 3\lambda I_0^\beta \right), \quad (16)$$

where we have used

$$\frac{d\bar{\Omega}}{d\varphi_0} = \frac{\varphi_0 3\lambda}{\bar{\Omega} \left(1 + \frac{3\lambda}{2} I_0^\beta \right)}. \quad (17)$$

The condition for a stationary point of (16) away from the origin is given by

$$\lambda \varphi_0^2 = \frac{m_b^2}{2} - 3\lambda I_0^\beta. \quad (18)$$

Using (14) in (18) we get (writing $\bar{\Omega}_s$ for the value of $\bar{\Omega}$ at the stationary point)

$$\bar{\Omega}_s^2 = 2\lambda \varphi_0^2. \quad (18a)$$

We use the suffix s instead of zero as the minimum occurs away from zero. If $(d^2V/d\varphi^2)|_{\varphi_0=0}$ is negative, $\varphi_0=0$ is not a minimum of V_G , and defining $\bar{\Omega}_s^2$ in terms of the minimum away from the origin we get from (18a) and (14) (writing $\bar{\Omega}_s^2 = M^2$)

$$M^2 = m_b^2 - 6\lambda I_0^\beta(M). \quad (19)$$

We can write (19) as

$$M^2 = m_b^2 - 6\lambda [\bar{I}_0^\beta + I_0(M) - I_0(m_R) + I_0(m_R)],$$

$I_0(M)$ being the temperature independent part of $I_0^\beta(M)$ and $\bar{I}_0^\beta(M)$ being the temperature dependent part. Following Stevenson [7] one can write (in (1 + 1) dimension)

$$I_0(M) - I_0(m_R) = -\frac{\ln M^2/m_R^2}{4\pi}. \quad (20)$$

We normalize the bare mass m_b in the following way:

$$m_R^2 = m_b^2 - 6\lambda I_0(m_R). \quad (21)$$

(Note that here we differ from the procedure of Su et al. who neglected the term given in (20).) Then we can write (19) as

$$M^2 = m_R^2 - 6\lambda \bar{I}_0^\beta(M) + 3\lambda \frac{\ln M^2/m_R^2}{2\pi}. \quad (22)$$

(Since $I_{-1}(M)$ is finite in (1 + 1) dimension, the normalization of λ is trivial and involves multiplication by a finite constant only, unlike in the (3 + 1) case. Hence we keep λ as it is.) From the appendix we get

$$\bar{I}_0^\beta(M) = \frac{1}{\pi} \int_0^\infty \frac{dk}{(k^2 + M^2)^{1/2} \left(\exp\left(\frac{k^2 + M^2}{T^2}\right)^{1/2} - 1 \right)}. \quad (23)$$

Using the following approximation for \bar{I}_0^β for large T , viz.

$$\bar{I}_0^\beta = \frac{1}{\pi} \left[\frac{\pi T}{M} + \frac{1}{2} \ln \left(\frac{M}{4\pi T} \right) + \frac{\gamma}{2} \right] + O\left(\frac{M^2}{T^2}\right), \quad (24)$$

we get from (22)

$$M^2 = m_R^2 - 3\lambda \frac{T}{M} - 3\frac{\lambda}{\pi} \left[\ln \left(\frac{M}{4\pi T} \right) + \gamma \right] + \frac{3\lambda \ln M^2/m_R^2}{2\pi}. \quad (25)$$

Su et al. calculated analytically the critical temperature, assuming $(3\lambda/\pi)(\ln(M/4\pi T) + \gamma)$ to be constant. However, the assumption is only valid when $\ln(M/4\pi T)$ is slowly varying with respect to T and hence does not hold for all values of T . Since we shall calculate T_c numerically, we shall not make any such assumptions. Solving (25) numerically we found (we take $\lambda=0.1$ for an illustrative case) that for $T > T_c = 1.77$ (in units of m_R) there is no real solution for M . The approximation solution of Su et al. gives $T_c \cong 1.69$. The discrepancy arises because Su et al. neglected the logarithmic terms. What happens is the following. When $T > T_c$, the only minimum of $\bar{V}^G(\varphi_0, \bar{\Omega})$ occurs at $\varphi_0 = 0$ and the definition of M changes to

$$\bar{\Omega}_0^2 = M^2 = \frac{d^2 \bar{V}^G(\varphi_0, \bar{\Omega})}{d\varphi_0^2} \Big|_{\varphi_0=0}. \quad (26)$$

Then from (16), differentiating $dV^G(\varphi_0, \bar{\Omega})/d\varphi_0$ again with respect to φ_0 and using the results

$$\frac{d\bar{\Omega}}{d\varphi_0} = \frac{\varphi_0}{\bar{\Omega}} \frac{3\lambda}{\left(1 + \frac{3\lambda}{2} I_{-1}^\beta\right)}, \quad (27)$$

we get

$$\frac{d^2 \bar{V}^G}{d\varphi_0^2} \Big|_{\varphi_0=0} = -\frac{m^2}{2} + 3\lambda \bar{I}_0^\beta(\bar{\Omega}_0). \quad (28)$$

Hence

$$M^2 = -\frac{m^2}{2} + 3\lambda \bar{I}_0^\beta(M) \simeq -\frac{m^2}{2} + \frac{3\lambda T}{M}, \quad (29)$$

a result obtained by Su et al. and also by Hazi and Stevenson [14] and Roditi [11] (their m^2 has to be replaced by $-m^2/2$). However Roditi has not discussed the fact that below T_c , the effective mass should be given by (19). In Fig. 1 we have plotted M^2 against T for $\lambda=0.1$. It shows a discontinuity at $T = T_c = 1.77$.

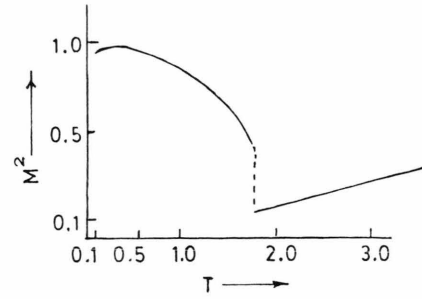


Fig. 1. Plot of M^2 against temperature T .

3. Calculation of $\bar{V}^G(\varphi_0, \bar{\Omega})$

To show what happens in $\bar{V}^G(\varphi_0, \bar{\Omega})$ for $T < T_c$ to $T > T_c$ we first regularize $\bar{V}^G(\varphi_0, \bar{\Omega})$. As it stands, the left hand side of (10) contains divergent integrals like $I_1(\bar{\Omega})$ and $I_0(\bar{\Omega})$. The regularization is achieved in the following way: From (10) we get

$$\begin{aligned} \bar{V}^G(\varphi_0, \bar{\Omega}) = & \bar{I}_1^\beta - \frac{\bar{\Omega}^2}{2} \bar{I}_0^\beta + I_1(\bar{\Omega}) - \frac{\bar{\Omega}^2}{2} \Delta I_0 - \frac{\bar{\Omega}^2}{2} I_0(m_R) \\ & + \frac{\lambda}{4} (\varphi_0^4 + 6\varphi_0^2 \bar{I}_0^\beta + 3\bar{I}_0^{\beta^2}) \\ & + \frac{\lambda}{4} [6\varphi_0^2 (\Delta I_0 + I_0(m_R)) + 6\bar{I}_0^\beta (\Delta I_0 + I_0(m_R)) \\ & + 3\Delta I_0^2 + 3I_0(m_R)^2 + 6I_0(m_R) \Delta I_0] \\ & - \frac{m_R^2}{4} (\varphi_0^2 + \bar{I}_0^\beta + \Delta I_0 + I_0(m_R)), \end{aligned} \quad (30)$$

where $\Delta I_0 \equiv I_1(\bar{\Omega}) - I_1(m_R)$.

Hence

$$\begin{aligned} V^G(\varphi_0, \bar{\Omega}) = & \bar{I}_0^\beta - \frac{\bar{\Omega}^2}{2} \bar{I}_0^\beta - m_R^2 L_2(x)/8\pi \\ & + \frac{\lambda}{4} [\varphi_0^4 + 6\varphi_0^2 \bar{I}_0^\beta + 3\bar{I}_0^{\beta^2} + 6\varphi_0^2 \Delta I_0 + 6\bar{I}_0^\beta \Delta I_0 + 3\Delta I_0^2] \\ & - \frac{m_R^2}{4} (\varphi_0^2 + \bar{I}_0^\beta + \Delta I_0) + D, \end{aligned} \quad (31)$$

where we took $\bar{\Omega}$ the effective mass, given in (14), and used the results

$$I_1(\bar{\Omega}^2) - I_1(m_R) = \frac{(\bar{\Omega}^2 - m_R^2)}{2} I_0(m_R) + \frac{L_2(x)}{8\pi}, \quad (32)$$

$$L_2(x) = x \ln x - x + 1, \quad (33)$$

$$x = \frac{\bar{\Omega}^2}{m_R^2} \quad (34)$$

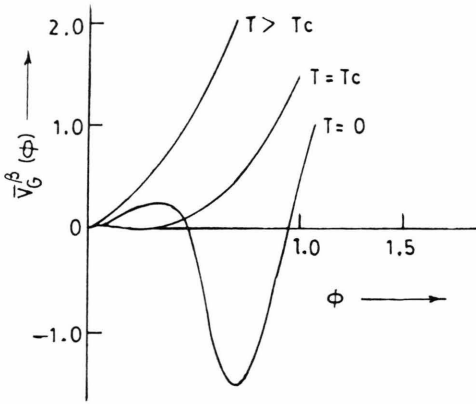


Fig. 2. Plot of $V_G^\beta(\phi)$ for different temperatures.

and m_R is given by (21). D is a divergent constant given by

$$D = I_1(m_R) - \frac{3}{4} m_R^2 I_0(m_R) - \frac{3}{4} \lambda I_0^2(m_R). \quad (35)$$

Finally we write

$$\bar{V}^G(\phi_0, \bar{\Omega}) = V^G(\phi_0, \bar{\Omega}) - D,$$

and it can be seen that $\bar{V}^G(\phi_0, \bar{\Omega})$ is finite. A schematic plot of $\bar{V}^G(\phi_0, \bar{\Omega})$ is given in Fig. 2 ($\lambda = 0.1$). It is obvious that for $T < T_c$ \bar{V}_G has a minimum at $\phi_0 = \bar{\phi}_0 \neq 0$. But when T approaches T_c , this minimum shifts towards 0, and for $T > T_c$ the only minimum is at $\phi_0 = 0$. The discontinuity of M as a function of T occurs because below and above T_c , M is defined differently.

4. Discussion and Conclusions

In this paper we have discussed soliton solutions at finite temperature using the Gaussian effective potential approach. The effective potential has a global minimum at a value of ϕ_0 away from zero when $T < T_c$. This structure disappears at $T \geq T_c$, the only minimum of \bar{V}_G occurs at $\phi_0 = 0$. Defining mass in different ways for $T < T_c$ and $T > T_c$, a mass gap equation has been obtained. Our result is different from that of Su et al. because their effective potential is really different from ours as they have not considered the contribution of kinetic terms in the Hamiltonian, as can be seen from the presence of \bar{I}_1^β in our \bar{V}^G term which is lacking in their expression for the “so called” effective potential. This term makes \bar{V}^G negative for large values of T . This type of behaviour of \bar{V}^G in finite temperature ϕ^4 theory has been observed recently by Okopinska [15] in the (3 + 1) dimension case. Though Roditi [11] has

discussed ϕ^4 finite temperature field theory, explicit results for $\bar{V}^G(\phi, \bar{\Omega})$ in (1 + 1) dimensional soliton theory was not given by him. Also we think the correct mass gap equation in (1 + 1) dimensional ϕ^4 theory has been presented here for the first time. Our method can be generalised easily to ϕ^6 and even the Sine-Gordon soliton theory, and work is in progress in this direction.

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Appendix

We write I_0^β in covariant form

$$I_0^\beta = \frac{i}{(2\pi)^2} \int \frac{d^2 k}{k^2 - \Omega^2 + i\epsilon}. \quad (A.1)$$

In the imaginary time mechanism, $\frac{1}{2\pi} \int dk_0$ is replaced by $T \sum_n$ and k_0 by $2\pi n$. Then from (A.1)

$$I_0^\beta = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \int \frac{dk}{(2\pi nT)^2 + k^2 + \Omega^2} \quad (A.2)$$

$$= \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \int \frac{dk}{X^2 + E^2}, \quad (A.3)$$

where $X = 2\pi nT$, $E = \sqrt{k^2 + \Omega^2}$.

Now we can write

$$\frac{1}{X^2 + E^2} = \left[\frac{1}{X - iE} - \frac{1}{X + iE} \right] \frac{1}{2iE}. \quad (A.4)$$

But

$$\frac{1}{X - iE} = \frac{1}{2\pi T} \sum_{n=0}^{\infty} \left(\frac{1}{n+l} - \frac{1}{n+l} \right) - \frac{i}{E}, \quad (A.5)$$

$$l = -iE/(2\pi T).$$

Writing

$$\zeta(s, l) = \sum_{n=0}^{\infty} \frac{1}{(n+l)^s},$$

where $\zeta(s, l)$ is the generalised ζ -function (Magnus et al. [16]), we obtain

$$\frac{1}{X - iE} = \frac{1}{2\pi T} \left[\frac{i}{y} + i\pi \coth \pi y \right] - \frac{i}{E}, \quad (A.6)$$

where $y = E/2\pi T$ and have used the property of the $\zeta(l, s)$ viz.,

$$\lim_{s \rightarrow 1} \left[\zeta(s, l) - \frac{1}{s-1} \right] = -\psi(l), \quad (\text{A.7})$$

where

$$\psi(l) = \frac{\Gamma'(l)}{\Gamma(l)} \quad (\text{A.8})$$

and $\Gamma(l)$ is the usual gamma function. Expanding $\coth \pi y$ in terms of $e^{-\pi y}$, we obtain after some straightforward calculation $I_0^\beta = I_0 + \bar{I}_0^\beta$, where I_0 is the temperature independent part, which is divergent inte-

gral* and \bar{I}_0^β is given by

$$\begin{aligned} \bar{I}_0^\beta &= \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dk}{\sqrt{k^2 + m^2}} \exp\{-nE/T\} \\ &= \frac{1}{n} \int_0^{\infty} \frac{dk}{\sqrt{k^2 + m^2}} \frac{1}{[\exp\{\pi\sqrt{(k^2 + m^2)^{1/2}/T}\} - 1]}. \end{aligned} \quad (\text{A.9})$$

* This is identical with the I_0 obtained by Stevenson [7] in the zero temperature case.

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